MIDTERM: Chapter 1
MATH 221C

Name: Jiajie (Jerry) Luo

In their first chapter, Guillemin and Pollack lay down the foundations of Differential Topology. Differential Topology deals with a class of objects called differentiable (smooth) manifolds, which are locally "like" Euclidean space. Much of the first chapter is based on maps between smooth manifolds, as well as when a subset of a manifold is a manifold. In order to solidify what any of this even means, we define a class of maps called smooth maps. Let $X \subset \mathbb{R}^n$ be any subset, and let $f : X \to \mathbb{R}^m$. If X were open, we already have a definition for what it means for f to be smooth - that it has partial derivatives of all order. But what if X was not open? In this case, partial derivatives aren't exactly well defined. To remedy this, we look to locally extending f. That is, for any $x \in X$, there is an open set $U \subset \mathbb{R}^n$ and a smooth map $F : U \to \mathbb{R}^m$ (in the open sense, as U is open) where $F|_{U \cap X} = f$. If $f : X \to Y$ was a smooth bijection with smooth inverse, we say that f is a diffeomorphism between X and Y. We can check that diffeomorphism is an equivalence relation on spaces.

With all this in place, we can finally define what a differentiable manifold is. We say that $X \subset \mathbb{R}^n$ is a k-manifold if it is locally diffeomorphic to an open set in $\mathbb{R}^k$. That is, for every $x \in X$, there is some (relatively) open $V \subset X$, an open set $U \subset \mathbb{R}^k$, and a map $\phi : U \to V$ such that ϕ is a diffeomorphism. We say ϕ is a local parameterization and ϕ^{-1} is a local coordinate system.

Now that we've set up the notion of a differentiable manifold, it makes sense to look at maps between them. As before, we have the notion of smooth maps, which we can restrict to the manifold setting. Since the map is smooth, it makes sense to look at the derivative of such maps. Let $X \subset \mathbb{R}^n$ be a k-manifold. While we can define the derivative in a way such that it sits inside its ambient space $\mathbb{R}^n$, since X is locally diffeomorphic to $\mathbb{R}^k$, it makes sense to look at it in a way that's more "independent" of its ambient space. To do this, let us take inspiration from multidimensional analysis. Suppose $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is open. We know that in this context, df_x is a linear map from $\mathbb{R}^m \to \mathbb{R}^n$. With this in mind, let us define the tangent space of X. Let $x \in X$ and $\phi : U \to X$ be a local parameterization of X at x, with $\phi(0) = x$. Since $U \subset \mathbb{R}^k$ is open, we can define $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^n$. We now define the tangent space of X at x to be $T_x(X) = d\phi_0(\mathbb{R}^k)$. Although it is not clear that this is a well defined space, in that it may depend on the choice of parameterizations, we can check that fortunately, the tangent space is invariant under choice of parameterization. This is good, because now, we have a platform to define the derivative of smooth maps of manifolds. The tangent space is the best vector-space approximation to X at x. In particular, the tangent space of a vector space about any point is that vector space itself. Moreover, we can check that $\dim(T_x(X)) = k$.

So, given $f : X \to Y$ where X is a k-manifold and Y is a l-manifold, let us define df_x. Given some $x \in X$, let $\phi : U \to X$ and $\psi : V \to Y$ be local parameterizations such that $\phi(0) = x$, $\psi(0) = f(x)$. We can choose U and V such that the following diagram commutes, with

$h = \psi^{-1} \circ f \circ \phi$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi \uparrow & & \uparrow \psi \\ U & \xrightarrow{h} & V \end{array}$$

Since $h : U \to \mathbb{R}^l$, we can define dh_0 in the standard way. From here, we can define df_x as the map that makes the following diagram commute:

$$\begin{array}{ccc} T_x(X) & \xrightarrow{df_x} & T_{f(x)}(Y) \\ d\phi_0 \uparrow & & \uparrow d\psi_0 \\ \mathbb{R}^k & \xrightarrow{dh_0} & \mathbb{R}^l \end{array}$$

As before, we can check that this definition is invariant under the choice of parameterization. In fact, df_x acts the way we would expect to. For example, the chain rule also acts as expected, in that $d(g \circ f)_x = dg_{f(x)} \circ df_x$.

Now that we've defined derivative maps and tangent spaces, let us focus on them specifically. In particular, let us examine how the properties of the derivative map tell us about the behavior of the map itself. One of the most important results in this topic is the inverse function theorem. The inverse function theorem states that given $f : X \to Y$ where df_x is an isomorphism of tangent spaces, f is a local diffeomorphism at x. In fact, we can reformulate this to mean that if df_x is an isomorphism, f is locally equivalent to the identity. That is, we can find a $U \subset \mathbb{R}^k$ and maps $\phi : U \to X$ and $\psi : U \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi \uparrow & & \uparrow \psi \\ U & \xrightarrow{id} & U \end{array}$$

This is an extraordinary result, in that the an isomorphism of tangent spaces tells us that locally around x and $f(x)$, X and Y are essentially "the same". Clearly, this can only work when $\dim(X) = \dim(Y)$. So what if $\dim(X) < \dim(Y)$? Here, the closest we can get to the condition we had for the inverse function theorem is to have df_x to be injective. We say that f is an immersion at x if df_x is an injection, and we say f is an immersion at $M \subset X$ if f is an immersion for every $x \in M$. As expected, we have a similar result to that of the inverse function theorem, which we call the local immersion theorem. The local immersion theorem tells us that when f is an immersion, it is locally equivalent to the canonical immersion (ie. the map that appends 0's). This too is an amazing result, in that if df_x is injective, then a neighborhood of X containing x is basically "a component" of a neighborhood of Y containing $f(x)$. Now, if $\dim(X) > \dim(Y)$, we say f is a submersion if df_x is surjective. Likewise, we have the local submersion theorem that tells us that if f is a submersion at x, then it is locally equivalent to the local submersion (ie. "chopping off components") at x. In a way, this tells us the "opposite" story, that locally, a neighborhood of Y containing $f(x)$ is "a component of" a neighborhood of X containing x.

It turns out that immersions and submersions are helpful in finding submanifolds! Let $f : X \to Y$ be an immersion. As one may expect, it is possible for $f(X)$ to be a submanifold of Y. But what exactly are sufficient conditions in order for $f(X)$ to be a manifold? It turns out that if f is an embedding, that is, an injective proper (preimage of closed sets are closed) immersion, then $f(X)$ is an immersion. Guilleman and Pollack offers a few examples of $f(X)$ not being a manifold when these conditions are not met. In a way, this is a surprise. After all, it's natural to think that f simply being an injective immersion should guarantee $f(X)$ to be a manifold. However, it turns out that there are always some strange pathological counterexamples to make this not the case!

Likewise, submersions are also helpful in telling the story of submanifolds. Let $f : X \to Y$. We say that $y \in Y$ is a regular value of f if for every $x \in f^{-1}(y)$, f is a submersion at x. Similarly, $x \in X$ is a regular point if f is a submersion at x, and otherwise, we say x is a critical point. Otherwise, y is a critical value. This leads us to a new way to determine submanifolds of X, that is, if $y \in Y$ is a regular value of f, $f^{-1}(y)$ is a submanifold of X, with dimension $\dim(X) - \dim(Y)$. We see that this is the first step determine whether or not the preimage of a submanifold is a manifold, by starting out with singleton sets, which leads us to a more general question: given $f : X \to Y$, and a submanifold $Z \subset Y$, when is $f^{-1}(Z)$ a submanifold of X? The answer to this question lies with the concept of transversality.

Given a map between manifolds $f : X \to Y$ and a submanifold $Z \subset Y$, we say that f is transverse to Z if for every $x \in f^{-1}(Z)$, $Image(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$. This leads to another result, that if f is transverse to Z, then $f^{-1}(Z)$ is a submanifold of X, with the codimension of $f^{-1}(Z)$ in X equal to the codimension of Y in Z. The idea of transversality is quite powerful, in that it provides the groundwork for "pulling back" a manifold into another manifold. It's also quite intuitive, in that in order for the preimage of a manifold to be another manifold, it must have the necessary structure at every point. Transversality gives us this structure, in that the preimage must "span enough space" in the "right directions".

Another useful idea from transversality is when the intersection of two submanifolds is a submanifold. We can define transversality of two spaces $X, Z \subset Y$ similar to how we did so above: given a submanifold $X \subset Y$, we say X and Z are transversal, and thus $X \cap Z$ is a manifold, if $T_x(X) + T_x(Z) = T_z(Y)$, for all $x \in X \cap Z$. We note that this is consistent with the definition above, by considering the inclusion map $i : X \hookrightarrow Y$, in which case, $i^{-1}(Z) \cap X = X \cap Z$, which is the definition of i being transverse to Z. Similar to the above, in order for the intersection to "span enough space", the tangent spaces must not be "the same". It turns out that the condition of transversality is stable, in that it is "invariant" under "slight perturbations". More concretely, for smooth manifolds this means that given f being transverse to Z, if F is a smooth homotopy of f to another function g, then there's some $\epsilon > 0$ such that for all $t < \epsilon$, f_t is transverse to Z. The idea of stability is discussed in more detail in the book, but due to the fact that it seems slightly less relevant in the grand scheme of what's going on, we will skip over these details.

From all of this, we see how the powerful and potent the idea of a submersion is. Simply by checking if a map has a surjective derivative at a point, we can determine quite a lot of structure between the spaces in question. That is, we see how submanifolds of the codomain can help determine submanifolds of the domain. We will now introduce yet again, another powerful tool: Sard's theorem. Sard's theorem tells us that given a smooth map $f : X \to Y$, almost all points in Y are regular values. That is, the set of critical values of Y has measure zero. This is quite a surprise, in that it applies for *any* smooth map. In fact, this holds even if $\dim(X) < \dim(Y)$, in which case, we know that no points of X is regular. One of the consequences of this is that there can be no space filling curves in $f : S^1 \to Y$, where $\dim(Y) > 1$. So, Sard's theorem tells us that there are "many ways" to obtain submanifolds of the domain, since the set of critical values of X is measure zero. Moreover, if we have $\dim(X) < \dim(Y)$, it tells us that there is no way for a smooth $f : X \to Y$ to be bijective, since every point of X is a critical point, and the critical values have measure zero. This itself may not be too big of surprise, but it does help reinforce the structure that we know.

One of the areas in which Sard's theorem comes to play is in Morse Theory. Given $f : X \to Y$, if $x \in X$ is a critical point, we say that it is nondegenerate if its Hessian matrix is nonsingular. If a is a critical point of f, the Morse Lemma tells us that there is a local coordinate system around a such that $f(x) = f(a) + \sum\limits_{i,j} h_{ij} x_i x_j$, where h_{ij} is the ijth entry of the Hessian of f at a. This leads us to Morse functions, functions whose critical points are all nondegenerate. Morse functions are useful in telling us the local topology around critical points, using the Morse Lemma. A consequence of Sard's theorem is the abundance of Morse functions. Let $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^N$. Now, for $a \in \mathbb{R}^n$, we define $f_a = f(x) + a_1 x_1 + \cdots + a_n x_n$. In this picture, Sard's theorem tell us that for almost every $a \in \mathbb{R}^n$, f_a is a Morse function (ie. the set of $b \in \mathbb{R}^n$ in which this is not true has measure zero). This is again a very impressive result, in that it tells us Morse functions, which have nondegenerate critical points, are "basically everywhere". This opens up the field of Morse theory as a whole, which tell us even more about the local structure of manifolds around critical points.

Another consequence of Sard's Theorem is the result of the Whitney embedding theorem, which deals with the dimension of Euclidean space required in order for there to be an embedding of a k-manifold. In the beginning, when we were given a k-manifold, we always had it embedded in an ambient euclidean space. But just how big does that space have to be? Whitney's theorem tells us that any k-manifold can be embedded in $\mathbb{R}^{2k+1}$. In fact, it can be shown that any k-manifold can be embedded in $\mathbb{R}^{2k}$, although showing that is much more difficult. This gives us a concrete number of dimensions an ambient euclidean space must have in order to have a k-manifold embedded inside.

All of this comes to show what a powerful piece of machinery Sard's theorem is, in that it gives us results that at first glance, doesn't seem related to the statement of the theorem at all. Moreover, it gives us a picture of the importance of submersions in determining manifold structure. Given a manifold, not only do submersions tell us about the inner structure of that manifold (ie. telling us about submanifolds), submersions also tell us about the relation the manifold has with its ambient space. All of this sets up a canvas for us to explore the properties of manifolds as a whole, which includes both the substructure of manifolds, as well as how they behave in the surrounding space they live in.